INDUCED REPRESENTATIONS OF THE ONE DIMENSIONAL QUANTUM GALILEI GROUP.

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Abstract. We study the representations of the quantum Galilei group by a suitable generalization of the Kirillov method on spaces of non commutative functions. On these spaces we determine a quasi-invariant measure with respect to the action of the quantum group by which we discuss unitary and irreducible representations. The latter are equivalent to representations on ℓ^2, *i.e.* on the space of square summable functions on a one dimensional lattice.

1. In some past works we used the quantum Galilei group $\mathcal{U}_q(G(1))$ to describe the kinematical symmetry of a spin chain on a one dimensional lattice [1]. The definition of its algebra follows from the relations of the subgroup of the Heisenberg group with discrete space translations and from the addition of a generator for time translations, *i.e.* the energy. It turns out that the eigenvalue equation for the Casimir of this algebra is the Schrödinger equation for the free particle on a one dimensional lattice. The physical requirement for the energy to be additive produces as a consequence a non cocommutative coalgebra and, by duality, a non commutative algebra for the representative functions. The deformation parameter of the quantum group thereby obtained has the interpretation of a lattice spacing, whose vanishing reproduces the Galilei Lie group $G(1)$ as symmetry of the continuum. The main purpose of this letter is to make evident the connection between discreteness and non commutativity in a physically meaningful example, by studying the representations of the quantum Galilei group in their natural framework of non commutative spaces. This method has already proved to be quite suitable for the study of harmonic analysis and special functions on homogeneous spaces [2]. We shall extend in the quantum group framework the technique of induced representations that gives a complete description of unitary representations for nilpotent Lie groups according to the Kirillov theory. For a meaningful physical interpretation of the algebra, an involution with non standard properties is required: this, in turn, implies a multiplicative involution on the space of quantized functions where we are going to represent the quantum group. We shall first construct the actions on the space of formal series and later on we determine appropriate subspaces where we define a quasi-invariant measure that permits to discuss unitary and irreducible representations. We finally prove that the space obtained in this non commutative context is isometric with the Hilbert space of square summable functions on the integers (*i.e.* with ℓ^2) and we also remark that we can extend the Kirillov theorem on the equivalence of representations.

2. The quantum Galilei group can be obtained by deforming the algebra $\mathcal{F}$ of the functions on $G(1)$ by means of a non trivial 1-cocycle η with values in $\bigwedge^2 Lie(G(1))$ that defines a Lie-Poisson structure. Its description is as follows. Let $\{\mu, x, t, v\}$ be a set of canonical coordinates of the second kind on $G(1)$; the fundamental fields on the group manifold corresponding to the generators of $Lie(G(1))$ are

$$\mathcal{X}_M = i\,\frac{\partial}{\partial\mu}\,, \qquad \mathcal{X}_P = i\,\frac{\partial}{\partial x}\,,$$
$$\mathcal{X}_T = i\,\frac{\partial}{\partial t}\,, \qquad \mathcal{X}_B = ix\,\frac{\partial}{\partial\mu} + it\,\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial v}\,.$$

We have:

(2.1) Proposition. *Let* $g \equiv (\mu, x, t, v) \in G(1)$. *The map* $\eta : G(1) \to \bigwedge^2 Lie(G(1))$ *given by*

$$\eta(g) = (2a\mu - 2axv + atv^2)\,\mathcal{X}_M \wedge \mathcal{X}_P - av^2\,\mathcal{X}_M \wedge \mathcal{X}_B - 2av\,\mathcal{X}_P \wedge \mathcal{X}_B$$

satisfies $\eta(gg') = \eta(g) + \mathrm{Ad}_g\eta(g')$ *and therefore is a (non trivial) 1-cocycle that defines the Lie-Poisson brackets*

$$\{\mu, x\} = -2a\mu\,, \qquad \{\mu, v\} = av^2\,, \qquad \{x, v\} = 2av\,, \qquad \{t, \cdot\} = 0\,. \qquad \blacksquare$$

Denote by $\mathcal{F}_q := \mathcal{F}_q(G(1))$ the Hopf algebra generated by $\{\mu, x, t, v\}$, whose relations are

$$[\mu, x] = -2a\mu\,, \qquad [\mu, v] = av^2\,, \qquad [x, v] = 2av\,,$$

t being a central element. Then $\mathcal{F}_q$ is a deformation of $\mathcal{F}$ in the direction of the Poisson brackets (2.1) and we obtain a Hopf algebra if we define the same comultiplication and antipode as in the classical case:

$$\begin{aligned} \Delta\mu &= \mu\otimes\mathbf{1}+\mathbf{1}\otimes\mu+v\otimes x+(1/2)v^2\otimes t\,, & \Delta t &= t\otimes\mathbf{1}+\mathbf{1}\otimes t\,,\\ \Delta x &= x\otimes\mathbf{1}+\mathbf{1}\otimes x+v\otimes t\,, & \Delta v &= v\otimes\mathbf{1}+\mathbf{1}\otimes v\,. \end{aligned}$$

$$S(\mu)=-\mu+vx-(1/2)v^2t\,,\qquad S(x)=-x+tv\,,\qquad S(t)=-t\,,\qquad S(v)=-v\,.$$

The quantum enveloping algebra $\mathcal{U}_q := \mathcal{U}_q(G(1))$ is expressed in terms of the generators $\{M, K, K^{-1}, T, B\}$ with relations

$$[K,T]=0\,,\qquad K\,B\,K^{-1}=B+aM\,,\qquad [B,T]=\frac{K-K^{-1}}{2a}\,,$$

where M is central and $KK^{-1}=K^{-1}K=1$. The Casimir reads

$$C = MT + \frac{1}{2\,a^2}\,[K+K^{-1}-2]\ . \tag{2.2}$$

The coalgebra structure is given by

$$\begin{aligned} \Delta M &= M\otimes K+K^{-1}\otimes M\,, & \Delta K &= K\otimes K\,,\\ \Delta T &= T\otimes\mathbf{1}+\mathbf{1}\otimes T\,, & \Delta B &= B\otimes K+K^{-1}\otimes B \end{aligned}$$

and

$$S(M)=-M\,,\qquad S(K)=K^{-1}\,,\qquad S(T)=-T\,,\qquad S(B)=-B-aM\,.$$

There is a nondegenerate duality pairing between $\mathcal{F}_q$ and $\mathcal{U}_q$ given by

$$\langle \mu^\alpha x^\beta t^\gamma v^\delta\,, I^{\alpha'}K^\ell T^{\gamma'}N^{\delta'}\rangle = i^{\alpha+\beta+\gamma+\delta}\ \alpha!\,\gamma!\,\delta!\,(a\,\ell)^\beta\ \delta_{\alpha,\alpha'}\,\delta_{\gamma,\gamma'}\,\delta_{\delta,\delta'}$$

where $I=K^{-1}M$, $N=KB$ and $\ell\in\mathbf{Z}$ while the other indices are in $\mathbf{N}$.

To define the classical limit we write $K=e^{iaP}$ and the Casimir (2.2) for $a\to 0$ reproduces the classical quadratic form $C_{\rm cl}=MT-P^2/2M$.

In order to have a meaningful physical interpretation of $\mathcal{U}_q$ as a deformation of the classical symmetry, we shall define the following involution $M^*=M$, $K^*=K^{-1}$, $T^*=T$, $B^*=B$. The compatibility with the Hopf structure results then in

$$(XY)^*=Y^*\,X^*\,,\qquad \Delta(X^*)=(*\otimes *)\circ\tau\circ\Delta X\,,$$

where $\tau(X \otimes Y) = Y \otimes X$. In the classification of $*$-structures given in [3], this is referred to as to the case (1,1). For this type of involution we have $* \circ S = S \circ *$ and we see that $(* \circ S)^2$ is different from identity. Therefore the use of the duality relations and of the operator $* \circ S$ for determining an involution on $\mathcal{F}_q$ requires some modifications. A direct calculation shows:

(2.3) Lemma. *Let $Q : \mathcal{U}_q \to \mathcal{U}_q$ be the morphism defined on the generators by*

$$Q(M) = M\,, \qquad Q(K) = K\,, \qquad Q(T) = T\,, \qquad Q(B) = B + aM\,.$$

Then Q is invertible and $Q^2 = (* \circ S)^2$. Therefore $\sigma = Q^{-1} \circ * \circ S$ is an antilinear multiplicative map of $\mathcal{U}_q$ with $\sigma^2 = \mathrm{id}$. Moreover $\Delta \circ \sigma = (\sigma \otimes \sigma) \circ \Delta$. ■

The involution on $\mathcal{F}_q$, again denoted by $*$, can now be defined as follows.

(2.4) Proposition. *Let f^* be defined by the relation*

$$\langle\, f^*, X \,\rangle = \overline{\langle\, f, \sigma(X) \,\rangle}\,.$$

Then the map $f \mapsto f^*$ is an involution on $\mathcal{F}_q$.

The generators $\{\mu, x, t, v\}$ are real under $*$ and we have:

(i) $(f_1 f_2)^* = f_1^* f_2^*$;

(ii) $\Delta(f^*) = (* \otimes *)\, \Delta f$.

According to [3] *the involution is therefore of the* $(0,0)$*-type.*

Proof. Since σ and complex conjugation are commuting involutions, it is immediate to see that also $f \mapsto f^*$ is an involution.

For the point (i) we have:

$$\begin{aligned}\langle (f_1 f_2)^*, X \rangle &= \overline{\langle f_1 \otimes f_2, \Delta \circ \sigma(X) \rangle} \\ &= \sum_{(X)} \overline{\langle f_1 \otimes f_2, \sigma(X_{(1)}) \otimes \sigma(X_{(2)}) \rangle} = \langle f_1^* f_2^*, X \rangle\end{aligned}$$

For the second item:

$$\begin{aligned}\langle \Delta f^*, X \otimes Y \rangle &= \overline{\langle f, \sigma(XY) \rangle} \\ &= \sum_{(f)} \overline{\langle f_{(1)}, \sigma(X) \rangle}\; \overline{\langle f_{(2)}, \sigma(Y) \rangle} = \langle (* \otimes *) \Delta f, X \otimes Y \rangle\end{aligned}$$

Finally, as $\sigma(I^\alpha K^\ell T^\gamma N^\delta) = (-)^{\alpha+\gamma+\delta}\, I^\alpha K^\ell T^\gamma N^\delta$, the generators of $\mathcal{F}_q$ are real. ■

3. In this section we present some algebraic aspects of induced representations that will be useful for applications to the quantum Galilei group. We take advantage from the fact that the undeformed group $G(1)$ is nilpotent, so that its representations are completely described by Kirillov theory [4]. In particular, due to the existence of maximal polarizing subalgebras, any irreducible representation is always induced from a character ω_J of a subgroup J. The representative space H_ω is the space of complex valued functions that are ω_J-covariant along the J-cosets,

$$f(jg) = \omega_J(j) f(g) \,, \qquad\qquad j \in J \,,\, g \in G \,, \tag{3.1}$$

and square integrable with respect to an invariant measure ν on the homogeneous space $J\backslash G$. Therefore H_ω is isometric to $L^2(J\backslash G, \nu)$ and $\rho_\omega = \mathrm{Ind}(J\!\uparrow G(1),\, \omega_J)$ reduces to the restriction to H_ω of the regular representation

$$\Big(\rho_\omega(g_2) f\Big)(g_1) = f(g_1 g_2) \,, \qquad\qquad g_1, g_2 \in G \,. \tag{3.2}$$

We now reformulate this scheme in a Hopf algebra framework. Let us examine first the covariance condition (3.1). Observe that $\mathcal{F}(J)$ is a Hopf algebra and $\pi_J : \mathcal{F}(G) \to \mathcal{F}(J)$, defined by $\pi_J(f) = f|_J$, is a Hopf algebra homomorphism. The character ω_J determines a corepresentation $1 \to \omega_J \,:\, \mathbf{C} \to \mathbf{C} \otimes \mathcal{F}(J) \simeq \mathcal{F}(J)$. Consider the right action of $j \in J$ on $f \in \mathcal{F}$ given by $(f \cdot j)(g) = f(jg)$ and let $\Lambda : \mathcal{F}(G) \to \mathcal{F}(J) \otimes \mathcal{F}(G)$ be the corresponding left coaction. The actual form of Λ is given by the standard dualization procedure, namely $\Lambda = (\pi_J \otimes \mathrm{id}) \circ \Delta$. The space H_ω of the induced representation will be defined as the subspace of functions in $\mathcal{F}(G)$ that satisfy the equivariance condition (3.1) rewritten as

$$\Lambda(f) = \omega_J \otimes f \ , \qquad\qquad f \in \mathcal{F}(G) \ . \tag{3.3}$$

On H_ω the comultiplication Δ determines a corepresentation Ψ_ω.

The generalization of the procedure to quantum groups is straightforward when a quantum subgroup exists, as in this case. We define the Hopf algebra $\mathcal{F}_q(J)$ generated by the three primitive elements $\{\widehat{\mu}, \widehat{x}, \widehat{t}\}$ with relations $[\widehat{\mu}, \widehat{x}] = -2a\widehat{\mu}$ and $[\widehat{t}, \cdot] = 0$. We also assume that the involution is of the type $(0,0)$ and that these three generators are real. The map $\pi_J : \mathcal{F}_q \to \mathcal{F}_q(J)$, defined on the generators by $\pi_J(\mu) = \widehat{\mu}$, $\pi_J(x) = \widehat{x}$, $\pi_J(t) = \widehat{t}$, $\pi_J(v) = 0$, is a surjective $*$-Hopf morphism. It is easy to verify that $\omega_{m,u} = \exp[-i(m\widehat{\mu} + u\widehat{t})]$ defines a one dimensional corepresentation of $\mathcal{F}_q(J)$.

(3.4) PROPOSITION. *Let* $\mathcal{H}_{m,u} = \Big\{\phi_{m,u}\, f(v)\Big\}$, *where* $\phi_{m,u} = \exp[-i(m\mu + ut)]$ *and* f *is a formal series in* v.

(i) *Each element* $\phi_{m,u}\, f(v) \in \mathcal{H}_{m,u}$ *is equivariant according to* (3.3)*;*

(ii) *on* $\mathcal{H}_{m,u}$ *we can define a coaction by*

$$\Psi_{m,u}\Big(\phi_{m,u}\, f(v)\Big) = \exp[-im(\mu\otimes\mathbf{1}+\mathbf{1}\otimes\mu+v\otimes x+(1/2)\,v^2\otimes t)] \\ \cdot\, \exp[-iu(t\otimes\mathbf{1}+\mathbf{1}\otimes t)]\ f(v\otimes\mathbf{1}+\mathbf{1}\otimes v)\,.$$

The corresponding action of an element $X \in \mathcal{U}_q$ *is given by* $\rho_{m,u}(X) = (\mathrm{id}\otimes X)\Psi_{m,u}$*;*

(iii) *letting* $\rho^{\uparrow}_{m,u}(X) = \phi^{-1}_{m,u}\ \rho_{m,u}(X)\ \phi_{m,u}$*, with* $X \in \mathcal{U}_q$*, we have*

$$\rho^{\uparrow}_{m,u}(M)\, f(v) = m\, f(v)\,, \qquad \rho^{\uparrow}_{m,u}(T)\, f(v) = \Big[\frac{mv^2}{2(1-iamv)} + u\Big]\, f(v)\,,$$

$$\rho^{\uparrow}_{m,u}(K)\, f(v) = \frac{1}{1-iamv}\, f(v)\,, \qquad \rho^{\uparrow}_{m,u}(B)\, f(v) = i\,(1-iamv)\,\frac{\partial}{\partial v}\, f(v)\,.$$

For the proof we need the following simple technical results.

(3.5) Lemma. *For complex numbers* r, s *and natural number* n*, the following holds:*

(i) $e^{r(\mu\, -\, s\, v)} = e^{r\mu}\,(1+ar\,v)^{-s/a} = (1-ar\,v)^{s/a}\, e^{r\mu}\,;$

(ii) $e^{r(\mu\, -\, s\, v^2)} = e^{r\mu}\, e^{-(srv^2(1+arv)^{-1})}\,;$

(iii) $\mu^{n\,-\,1}\,(\mu+na\,v) = (\mu+a\,v)^{n}\,.$

Proof. *(i)* Suppose that $L(v)$, as a formal series of v, satisfies the relation $[L(v),\mu] = s\,v$. Then it is easily verified that

$$e^{r(\mu\, -\, s\, v)} = e^{-L(v)}\, e^{r\mu}\, e^{L(v)}$$

Therefore, since $\exp[-L(v)]\ \exp[r\mu] = \exp[r\mu]\ \exp[-L(v)]\,(1+ar\,v)^{-s/a}$, the final result follows by substituting this expression into the former one. With a similar method also item *(ii)* can be proved. The last statement follows by an easy induction. ∎

Proof of Proposition (3.4) The proof is a consequence of the duality relations, the properties of the comultiplication and of the results of Lemma (3.5). We report here only on some more meaningful points. For getting the action of the mass, a first computation shows that

$$(\mathrm{id}\otimes M)\,(\Delta\mu)^n = in\mu^{n-1}\,.$$

It is then straightforward to find the result.

Concerning K a useful relation turns out to be

$$(\mathrm{id} \otimes K)\, (\Delta\mu)^n = (\mu - av)^n\,,$$

as a consequence of which we find

$$\rho_{m,u}(K)\Big(\phi_{m,u}\, f(v)\Big) = e^{-im(\mu - a\, v)}\, e^{-iut}\, f(v)\,.$$

Using (3.5i) to factorize $\phi_{m,u}$ we find $\rho^{\uparrow}_{m,u}(K)$ as in the proposition.

In order to obtain the action of the energy, we point out that

$$(\mathrm{id} \otimes e^{irT})\, (\Delta\mu)^n = (\mu - rv^2/2)^n$$

and then, by differentiating with respect to r and using (3.5ii), we get the result. Finally, the boost does not present any particular problem. ∎

4. The framework of formal series is not suitable to discuss unitary representations. In order to determine an inner product, a quasi-invariant measure has to be defined on the representation space. We must therefore find appropriate subspaces of formal series that carry such a measure. Although we will not be able to define a coaction, we show that an action is well defined on these restrictions: this is sufficient to define representations that will prove to be unitary. Irreducibility is finally discussed.

(4.1) Lemma. (i) *The following holds*

$$(1 + iamv)\, \phi_{m,u}\, v\, \phi^{-1}_{m,u} = v\,;$$

(ii) *let* $v_k = (\phi_{m,u})^{-k}\, v\, (\phi_{m,u})^k$. *Then*

$$iam(k - \ell) v_k\, v_\ell = v_k - v_\ell\ . \qquad ∎$$

Consider the linear space freely generated over the set of finite words in $\phi_{m,u}$, $\phi^{-1}_{m,u}$, and v with relations (4.1i). Let H_0 be the subspace generated over the words whose total degree in $\phi_{m,u}$ is equal to that in $\phi^{-1}_{m,u}$.

(4.2) Proposition. *If* $H_0^{(k)} = v_k\, \mathbf{C}[v_k]$, *i.e. the space of polynomials in* v_k *with vanishing constant term, then*

$$H_0 = \mathbf{C} \oplus \Big(\bigoplus_{k=-\infty}^{\infty} H_0^{(k)} \Big)\,.$$

Proof. It is a direct consequence of the definitions and of relation (4.1*ii*). ∎

Let $H^{(k)}_{m,u} = \phi_{m,u}\, H^{(k)}_0$ and $H_{m,u} = \phi_{m,u}\, H_0$. Since the coproduct of $\phi_{m,u}$ is not polynomial in $\phi_{m,u}\,,\phi^{-1}_{m,u}\,,$ and v no coaction of the quantum Galilei group can be defined on H_0 and $H_{m,u}$. However the action obtained as in (3.4*ii*) admits a restriction to $H_{m,u}$ and therefore determines a representation of $\mathcal{U}_q$.

(4.3) Proposition. *$\mathcal{U}_q$ is represented on $H_{m,u}$ as follows:*

$$\begin{aligned}
&\rho_{m,u}(M)\,\phi_{m,u}\,v^n_k = m\phi_{m,u}\,v^n_k \ ,\\
&\rho_{m,u}(K)\,\phi_{m,u}\,v^n_k = \phi_{m,u}\,v^n_k(1+iamv_1)\ ,\\
&\rho_{m,u}(K^{-1})\,\phi_{m,u}\,v^n_k = \phi_{m,u}\,v^n_k(1-iamv_0)\ ,\\
&\rho_{m,u}(T)\,\phi_{m,u}\,v^n_k = \phi_{m,u}\,v^n_k\,(\frac{i}{2a}(v_0-v_1)+u)\ ,\\
&\rho_{m,u}(B)\,\phi_{m,u}\,v^n_k = i\,n\,\phi_{m,u}v^{n-1}_k\,(1-imv_0)(1+iam\,k\,v_k)^2\ .
\end{aligned}$$

The representation is reducible but not completely reducible. The only irreducible component is the restriction to

$$H^{\rm irr}_{m,u} = \mathbf{C}\oplus H^{(0)}_{m,u}\oplus H^{(1)}_{m,u}\,.$$

Proof. The proof is obtained by a direct calculation from (3.4*iii*) and (4.1*ii*). ∎

(4.4) Corollary. *Let $\chi_m = (1+iamv_1)\in \mathbf{C}\oplus H^{(1)}_0$. This element is invertible in H_0 with $\chi^{-1}_m = (1-iamv_0)\in\mathbf{C}\oplus H^{(0)}_0$. Then $\{\phi_{m,u}\,\chi^\ell_m\}_{\ell\in\mathbf{Z}}$ is a basis for $H^{\rm irr}_{m,u}$ and the irreducible representations given in (4.3) read*

$$\begin{aligned}
&\rho_{mu}(K^{\pm1})\,\phi_{m,u}\,\chi^\ell_m = \phi_{m,u}\,\chi^{\ell\pm1}_m\,, \qquad \rho_{mu}(B)\,\phi_{m,u}\,\chi^\ell_m = m\,\ell\,\phi_{m,u}\,\chi^\ell_m\,,\\
&\rho_{mu}(T)\,\phi_{m,u}\,\chi^\ell_m = \phi_{m,u}\,\chi^\ell_m\,(\frac{1}{2a^2m}\,(2-\chi_m-\chi^{-1}_m)+u)\,,
\end{aligned}$$

while $\rho_{mu}(M)$ is the multiplication by m . ∎

Let $H^{\rm irr}_0 = \mathbf{C}\oplus H^{(0)}_0\oplus H^{(1)}_0$. Define a linear functional $\nu_a : H^{\rm irr}_0\to\mathbf{C}$ by

$$\nu_a(1) = 1\ , \qquad \nu_a(v^n_0) = \frac{1}{(iam)^n}\ , \qquad \nu_a(v^n_1) = \frac{1}{(-iam)^n}\ , \tag{4.5}$$

so that $\nu_a(\chi^\ell_m) = \delta_{\ell,0}$.

(4.6) PROPOSITION. *The following holds:*

(*i*) *if* $a, b \in H^{\mathrm{irr}}_{m,u}$ *then* $a^*b \in H^{\mathrm{irr}}_0$ *and* $\langle a, b\rangle = \nu_a(a^*b)$ *defines a scalar product on* $H^{\mathrm{irr}}_{m,u}$ *;*

(*ii*) *the Hilbert space obtained by completing* $H^{\mathrm{irr}}_{m,u}$ *with respect to the scalar product defined in* (*i*) *is isometric to* $L^2(\,[0, 2\pi/a)\,, a\,dp/(2\pi))$ *and, by Fourier transform, to* ℓ^2*;*

(*iii*) *the representation in* (4.4) *is unitary.*

Proof. The first statement of (*i*) is a direct consequence of the relation $\phi^*_{m,u}\chi^*_m\phi_{m,u} = \chi^{-1}_m$. Moreover

$$\langle\phi_{m,u}\,\chi^n_m, \phi_{m,u}\,\chi^\ell_m\rangle = \nu_a(\phi^*_{m,u}(\chi^n_m)^*\phi_{m,u}\,\chi^\ell_m) = \nu_a(\chi^{\ell-n}_m) = \delta_{\ell,n} \ .$$

Thus

$$\langle\phi_{m,u}\,\chi^n_m, \phi_{m,u}\,\chi^\ell_m\rangle = \frac{a}{2\pi}\int_0^{2\pi/a} e^{iap(\ell-n)}dp\,.$$

We therefore obtain a natural isometry of H^{irr}_0 into $L^2(\,[0, 2\pi/a)\,, a\,dp/(2\pi))$ mapping χ^ℓ_m in plane waves.

Item (*iii*) is proved by verifying that $\rho_{m,u}(X^*) = \rho_{m,u}(X)^t$ on the generators. ∎

(4.7) COROLLARY. *The measure* ν_a *is quasi-invariant with respect to the restriction to* H^{irr}_0 *of the regular representation of* $\mathcal{U}_q$*. Indeed for each* $f \in H^{\mathrm{irr}}_0$ *and* $X \in \mathcal{U}_q$ *we have*

$$\nu_a((\mathrm{id}\otimes X)\Delta f) = \nu_a(f\,\xi(X))$$

where

$$\xi \mapsto \xi(X) = \sum_{n=0}^{\infty}(iam)^n\langle v^n, X\rangle\,\chi^n_m\ :\mathcal{U}_q \to H^{\mathrm{irr}}_0\,. \qquad ∎$$

(4.8) REMARKS. We shall conclude by pointing out some peculiar differences between the classical and the quantum situation and by describing the extension of the Kirillov theorem to the quantum Galilei group.

(*i*) The measure ν_a is finite on the coordinate v, at difference with the classical case where it diverges, as can be checked by taking the limit $a \to 0$. Noncommutativity means here regularization.

(*ii*) Contrary to the classical situation, ν_a is only quasi-invariant and becomes invariant only in the continuum limit, [5]. Nevertheless these induced representations are unitary. This feature is closely connected with the non standard properties of the involution.

(iii) On H_0^{irr} we have a scalar product defined by $\langle a, b\rangle = \nu_a(a^\dagger b)$ where $a^\dagger = \phi^*_{m,u} a^* \phi_{m,u}$. The Hilbert space obtained is isometric with the space given in (4.6). If we formally write $\chi_m = \sum_{k=0}^\infty (1-\chi_m^{-1})^k$, we note that this series doesn't converge in norm.

(iv) The classical limit of the representation (4.4) corresponds to the choice of the point $mM^* + uT^*$ on the coadjoint orbit $\mathcal{O}_{m,u} = \Big\{mM^* + \alpha P^* + (u + \alpha^2/2m)\, T^* + \beta B^* \,|\, \alpha, \beta \in \mathbf{R}\Big\}$, characterized by the mass m and by the "internal energy" u. A generic point of the orbit $\mathcal{O}_{m,u}$ gives a character $\exp[-i(m\widehat{\mu} + \alpha\widehat{x} + (u + \alpha^2/2m)\widehat{t}\,)]$ that determines a representation ρ on the space of equivariant functions $H = \{\phi\, f(v)\}$, where $\phi = \exp[-i(m\mu + \alpha x + (u + \alpha^2/2m)t)]$. This classical procedure extends to the quantum case. For each triple of numbers $\mathbf{c} = (c_1, c_2, c_3)$ we have that $\omega_{\mathbf{c}} = \exp[-ic_1\widehat{\mu}] \cdot \exp[-ic_2\widehat{t}\,] \cdot \exp[-ic_3\widehat{x}]$ defines a one dimensional corepresentation of $\mathcal{F}_q(J)$. The corresponding induced action $\rho^\uparrow = \phi_{\mathbf{c}}^{-1}\, \rho\, \phi_{\mathbf{c}}$ is

$$
\begin{aligned}
&\rho^\uparrow(M)\, f(v) = c_1 \exp(ia\, c_3)\, f(v)\,,\\
&\rho^\uparrow(K)\, f(v) = (\exp(-ia\, c_3) - iac_1 \exp(ia\, c_3)\, v)^{-1}\, f(v)\,,\\
&\rho^\uparrow(B)\, f(v) = i\,(\exp(-ia\, c_3) - ia\, c_1 \exp(ia\, c_3)\, v)\, \frac{\partial}{\partial v}\, f(v)\,,\\
&\rho^\uparrow(T)\, f(v) = \Big(\frac{c_1 \exp(4ia\, c_3)\, v^2}{2(1 - ia\, c_1 \exp(2ia\, c_3)\, v)} - \frac{1 - \exp(2ia\, c_3)}{2ia}\, v + c_2\Big)\, f(v)\,,
\end{aligned}
$$

with $\phi_{\mathbf{c}} = \exp[-ic_1\mu] \cdot \exp[-ic_2\, t] \cdot \exp[-ic_3\, x]$.

The calculation of the Casimir (2.2) gives

$$
C = c_1\, c_2\, e^{iac_3} + \frac{1}{a^2}(\cos(ac_3) - 1)\,,
$$

so that the equivalence with the representation in $(3.4iii)$ is proved posing

$$
c_1 = m\, e^{-iac_3} \qquad \text{and} \qquad c_2 = -\frac{1}{ma^2}(\cos(ac_3) - 1) + u\,.
$$

The construction of unitary representations requires that c_3 is real, so that $\omega_{\mathbf{c}}^* \omega_{\mathbf{c}} = 1$ and the explicit equivalence is easily obtained by defining the plane waves as $\chi_{m,c_3} = e^{iac_3} + iam\, \phi_{\mathbf{c}}^*\, v\, \phi_{\mathbf{c}}$ and $\chi_{m,c_3}^{-1} = e^{-iac_3} - iam\, v$.

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